Lecture 8. Basics of nodal curves.

- Normalization
- Line bundles on a nodal curve
- Dualizing sheaf.

So we go back to the basic (no family appears)

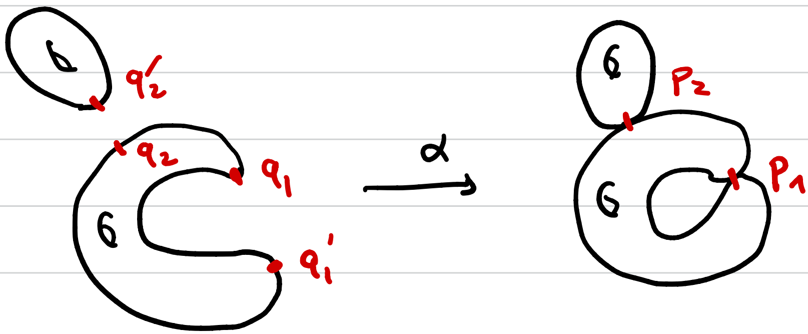
§ 1 Normalization.

Let C : connected nodal curve $/k$. In particular C is reduced.

We can always find a smooth birational model of C by taking normalization

$$\alpha : \tilde{C} \longrightarrow C$$

(i.e. $\operatorname{Spec} A \hookrightarrow C$ open, take the integral closure of A.
$\alpha|_{\operatorname{Spec} A} : \operatorname{Spec} \tilde{A} \longrightarrow \operatorname{Spec} A$)



Let $D = \{p_1, \cdots, p_\nu\} \subset C$: set of nodes
$\tilde{D} = \{q_1, q_1', \cdots q_\nu, q_\nu'\} = \alpha^{-1} D \subset \tilde{C}$.

Let $U = C \setminus D$ ← smooth locus of C.

The canonical map $\mathcal{O}_C \longrightarrow \alpha_* \alpha^* \mathcal{O}_C$ is injective.
Also it is an isomorphism on U.

$$0 \to \mathcal{O}_C \longrightarrow \alpha_* \mathcal{O}_{\tilde{C}} \longrightarrow \bigoplus_{p \in D} k_p \longrightarrow 0 \qquad \text{(Eq1)}$$

The morphism α is finite.
$\Rightarrow$ $\alpha_* : QCoh(\tilde{C}) \to QCoh(C)$ is exact &
$H^i(\tilde{C}, \mathcal{F}) \cong H^i(C, \alpha_* \mathcal{F})$.

<u>Exercise</u> (i) $\chi(\mathcal{O}_{\tilde{C}}) = \chi(\mathcal{O}_C) + \nu$
(ii) Let $\tilde{C}_i$: connected components of $\tilde{C}$.

$$P_a(C) = \sum_i P_a(\tilde{C}_i) + h^1(\Gamma_C) \leftarrow \text{genus formula!}$$

§2. Line bundles on a nodal curve.

· When we study alg varieties, the most basic question is to study $\mathrm{Pic}(C)$.

· When C = smooth, proj variety / k, we saw

$$\mathrm{Pic}^{\circ}(C) \simeq H^0(\Omega_C) / H_1(C, \mathbb{Z})$$

which is proper (even projective!) over k.

Q) What about a general nodal curve?

Similar exact seq also holds for $\mathcal{O}_C^{\times}$.

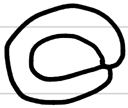
$$1 \to \mathcal{O}_C^{\times} \to \alpha_* \mathcal{O}_{\tilde{C}}^{\times} \to \prod_{p \in D} k_p^{\times} \to 1 \quad \text{(Eq2)}$$

multiplicative notion.

Taking long exact seq :

$$1 \to k^{\times} \to (k^{\times})^t \to (k^{\times})^{\omega} \to \mathrm{Pic}(C) \xrightarrow{\alpha^*} \mathrm{Pic}(\tilde{C}) \to 1$$

where t = number of connected components of $\tilde{C}$.

Example: $C =$  be the rational nodal curve

$\rightsquigarrow (\tilde{C}, \tilde{D}) = (\mathbb{P}^1, \{q, q'\})$

$\Rightarrow$ Choosing a line bdl on C

$=$ choosing a line bundle $\tilde{L}$ on $\mathbb{P}^1$ &

Isomorphism

$$\xi_p : \tilde{L}_q \xrightarrow{\sim} \tilde{L}_{q'} \qquad \rightsquigarrow \text{choice} = k^\times$$

Def Let L be a line bundle on C. The multi degree of L is the t-tuple

$$(d_1, --- d_t) = (\deg_{\tilde{C}_1}(\alpha^* L), --- , \deg_{\tilde{C}_t}(\alpha^* L)) \in \mathbb{Z}^t.$$

Let $\mathrm{Pic}^{\underline{0}}(C) \subset \mathrm{Pic}(C)$ be the subgroup of multi degree 0 line bundles on C

Exercise From (Eq 2), show that

$$1 \to (k^\times)^{\nu - t + 1} \longrightarrow \mathrm{Pic}^\circ(C) \longrightarrow \overset{\prod_{i=1}^{t} \mathrm{Pic}^\circ(\tilde{C}_i)}{\overset{\shortparallel}{\mathrm{Pic}^\circ(\tilde{C})}} \to 1$$

is exact. So if $h^1(\Gamma) = \nu - t + 1 > 0$, $\mathrm{Pic}^\circ(C)$ is not proper! /k

Cor Let L: line bundle on C. Then

$$\chi(L) = \underbrace{\deg L}_{= \sum d_i} + 1 - p_a(C)$$

pf) Take $\otimes L$ with (Eq 2). It is still exact. $\square$

§3. Dualizing sheaf of C

Ref: B. Conrad, Grothendieck duality & Base Change.

(3.1) Definitions

Def A dualizing sheaf for C is a coherent sheaf $\omega_{C/k}$ together with a trace map

$$tr : H^1(\omega_C) \longrightarrow k$$

s.t. for any $\mathcal{F} \in Coh(C)$

$$Hom(\mathcal{F}, \omega_C) \times H^1(\mathcal{F}) \longrightarrow H^1(\omega_C) \xrightarrow{tr} k$$

gives an isomorphism $Hom(\mathcal{F}, \omega_C) \simeq H^1(\mathcal{F})^\vee$.

Facts ① If it exists, it is unique.

② For proj variety, it always exists

③ For a reduced curve, more is true:

$$Ext^1(\mathcal{F}, \omega_C) \simeq H^0(\mathcal{F})^\vee.$$

If C: smooth, projective $/k$, $\omega_{C/k} \simeq \Omega_{C/k}$. sheaf of holo. differentials

For a nodal curve C, $j = U = C^{sm} \hookrightarrow C$,

$$j^* \omega_{C/k} \simeq \Omega_{U/k}$$

Q) What is the geometric meaning of $H^0(\omega_C)$?
A) (Rosenlicht) φ can be realized by a meromorphic differential on $\tilde{C}$ with the residue condition !!

Let's refine our question as follows:

<u>Lemma</u> $\omega_{C/k} \longrightarrow j_* j^* \omega_{C/k} = j_*(\Omega_{U/k})$

Proof) let $\mathcal{K} = \ker(\omega_{C/k} \longrightarrow j_* j^* \Omega_{C/k}) \subset \omega_{C/k}$ supported on $C \setminus U$. $\Rightarrow H^1(C, \mathcal{K}) = 0$. So

$$\mathrm{Hom}(\mathcal{K}, \omega_{C/k}) \simeq H^1(\mathcal{K})^\vee = 0 \quad \Rightarrow \quad \mathcal{K} = 0$$

$\boxtimes$

Let $\underline{\Omega}_{K/k}$: sheaf of meromorphic differentials on C (ie $K = k(C)$. $j_K : \mathrm{Spec}\, K \to C$ $\underline{\Omega}_{K/k} := j_* \Omega_{K/k}$).

So from the above lemma, $\omega_{C/k} \hookrightarrow j_* \Omega_{U/k} \hookrightarrow \underline{\Omega}_{K/k}$

Q') What is the image of $\omega_C \hookrightarrow \underline{\Omega}_{K/k}$?
What is the trace map ?

(3.2) Regular differentials

Let's go back to our normalization

$$\begin{tikzcd} \tilde{U} \arrow[r] \arrow[d, "\cong"'] & \tilde{C} \arrow[d, "\alpha"] & \tilde{D} \arrow[l, hook'] \arrow[d] \\ U \arrow[r, hook, "j"] & C & D \arrow[l, hook'] \end{tikzcd}$$

$\tilde{K} = \prod K_i$, $K_i = k(\tilde{C}_i)$

$\alpha_* \Omega_{\tilde{K}/k} \simeq \Omega_{K/k}$.

__Recall__. For any smooth proj curve X/k, we have the residue map : for a closed point $p \in X$,

$$\mathrm{res}_p : \Omega_{K/k} \longrightarrow k \qquad t \in \mathcal{O}_{X,p} \text{ unif}$$

$$\Big(\sum_{i \geq l} a_i t^i\Big) \longmapsto a_{-1}.$$

__Residue Theorem__. $\displaystyle\sum_{p \in X^0} \mathrm{res}_p(\eta) = 0$

X^0 = set of closed pts.

↳ Stoke's thm $k = \mathbb{C}$.

In fact we can write tr_C in terms of res_p (see Hartshorne).

Def A sheaf of regular differentials $\omega^{reg}_{C/k}$ is a subsheaf

$$\omega^{reg}_{C/k} \subset \alpha_* \Omega_{\tilde{C}/k}(\tilde{D})$$

s.t $\forall\ V \subset C$ open,

$$\omega^{reg}_{C/k}(V) = \left\{ \eta \in \Omega_{\tilde{C}/k}(\tilde{D})(\alpha^{-1}V) \;\middle|\; \begin{array}{l} p \in D, \\ \underbrace{res_q(\eta) + res_{q'}(\eta) = 0}_{\text{"residue matching condition"}} \end{array} \right\} \quad (*)$$

Check $\omega^{reg}_{C/k}$ has the constant rank 1. So in particular it is a line bundle on C.

• What is the reason to put condition $(*)$?

Let me try to justify (my understanding of) $(*)$:
Interesting things happen at each node., so let's focus around a node $p \in C$.

Étale locally, $C = V(F) \subset \mathbb{A}^2_{x,y}$ $F = xy$

$\hat{\mathcal{O}}_{C,p} \simeq k[\![x,y]\!]/(xy)$

We saw that if $C \overset{i}{\hookrightarrow} P$ ← sm of dim m regular embedding,

$$\omega_{C/k} \simeq i^* \Omega^m_P \otimes \Lambda^{top} \mathcal{N}_{C/P}.$$

In our case, $\omega_{C/k} \simeq \Omega^2_{\mathbb{A}^2/k}|_C \otimes \mathcal{O}_{\mathbb{A}^2}(F)|_C$ so

$$\widehat{\omega_{C/k,p}} \simeq \hat{\mathcal{O}}_{C,p} \left\langle \frac{dx \wedge dy}{F} \right\rangle$$

So $(*)$ should come from

$$0 = dF = x\,dy + y\,dx \implies \frac{dx}{x} = -\frac{dy}{y}.$$

$$\hat{\Omega}_{C,p} \doteq \hat{\mathcal{O}}_{C,p} \langle dx, dy \rangle / (x\,dy + y\,dx)$$

There is a map

$$\rho : \Omega_{C/k} \longrightarrow \omega_{C/k}$$

which induces an isomorphism at smooth points, on $p \in D$

$$\rho_p : \hat{\Omega}_{C,p} \longrightarrow \hat{\omega}_{C,p} \quad \alpha \longmapsto \tilde{\alpha} \wedge \frac{dF}{F} \ (\mathrm{mod}\ F)$$

any lift to $\hat{\Omega}_{\mathbb{A}^2,p}$

(well-define bc $dF \wedge dF = 0$). So

$$\rho_p(dx) = x\,\frac{dx \wedge dy}{F}, \quad \rho_p(dy) = -y\,\frac{dx \wedge dy}{F}$$

So one can "think" of $\hat{\omega}_{C,p}$ as

$$\hat{\omega}_{C,p} \simeq \hat{\mathcal{O}}_{C,p}\left\langle \frac{dx}{x}, \frac{dy}{y} \right\rangle \Big/ \left(\frac{dx}{x} + \frac{dy}{y} \right)$$

So in the normalization, $\mathrm{res}_q(\eta) + \mathrm{res}_{q'}(\eta) = 0$. ※

§ Trace map

Since $\omega^{reg}_{C/k} = \Omega_{K/k}$ at the generic pt of C,

$$0 \to \omega^{reg}_{C/k} \hookrightarrow \Omega_{K/k} \longrightarrow \bigoplus_{p \in C^\circ} i_{p*}\left(\Omega^1_{K/k.p} / \omega^{reg}_{C/k.p}\right) \to 0$$

C° ← set of closed pts.

$i_p : \mathrm{Spec}(\mathcal{O}_{C.p}) \longrightarrow X$

$$\rightsquigarrow \Omega_{K/k} \to \bigoplus_{p \in C^\circ} \Omega_{K/k.p} / \omega^{reg}_{C/k.p} \to H^1(C.\omega^{reg}_{C/k}) \to 0$$

For $p \in C^\circ$, we define

$$\mathrm{res}_p : \Omega_{K/k.p} \longrightarrow k, \quad \eta \longmapsto \sum_{q \in \alpha^{-1}(p)} \mathrm{res}_q(\eta)$$

By (*) res_p kills $\omega^{reg}_{C/k.p}$. Moreover, the composition

$$\Omega_{K/k} \longrightarrow \bigoplus_{p \in C^\circ} \Omega_{K/k.p} / \omega^{reg}_{C/k.p} \xrightarrow{\sum \mathrm{res}_p} k$$

is zero by the residue formula.

$$\Longrightarrow \mathrm{res}_C : H^1(\omega^{reg}_{C/k}) \longrightarrow k$$

Thm (Rosenlicht) Two coherent subsheaves

$$\omega_{C/k},\ \omega^{res}_{C/k} \subset j_*\Omega_{U/k}\ (\subseteq \underline{\Omega}_{K/k})$$

coincide. and $tr_{C/k} = res_{C/k}$ (upto sign)